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Vacuum stress tensor for a slightly squashed Einstein universe

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Abstract. The effect of the deviation from spherical symmetry on the effective Lagrangian of a scalar field is investigated.

1. Introduction

The system under study here is a free scalar field, massive and conformally coupled on an Einstein universe with a 'squashed' spatial section, i.e. on a frozen Mixmaster universe.

There are several reasons for choosing this space–time. Firstly it is not conformally flat. This enables us to study the massless case more generally. Secondly, it is soluble in perturbation expansion about the (exactly soluble) spherical problem (Dowker and Critchley 1977), and thirdly, deviations from spherical symmetry have been included in the calculations of Fischetti *et al* (1979) on the back reaction problem in the early universe.

The particular spatial section we discuss is the one that retains a certain amount of symmetry. The symmetry group of the three-sphere S^3 is $SU(2)\times SU(2)$, corresponding to body-fixed and space-fixed 'rotations' of the (ideal) spherical top whose configuration space is isometric to S^3. A 'symmetric' top (an oblate or prolate spheroid) has a configuration space whose continuous symmetry group is $SU(2)\times SO(2)$. This is the case in which we are interested, and the evaluation of the mode functions and energies is standard (Hu 1973).

Our limited aim is to obtain the effective Lagrangian and vacuum-averaged stress tensor. We begin the calculations by discussing the mode problem.

2. Modes and Green function

The metric is

$$(\mathrm{d}s)^2=(\mathrm{d}t)^2-l_1^2\,(\mathrm{d}\theta)^2-l_3^2\,(\mathrm{d}\psi)^2-2l_3^2\cos\theta\,\mathrm{d}\phi\,\mathrm{d}\psi-(\sin^2\theta l_1^2+l_3^2\cos^2\theta)\,\mathrm{d}\phi^2. \tag{1}$$

The conformally coupled Klein–Gordon equation

$$(\Box+\mu^2+R/6)\phi=0 \tag{2}$$

takes the form

$$\left(\frac{\partial^2}{\partial t^2}-\frac{1}{l_1^2}X_1^2-\frac{1}{l_1^2}\quad X_2^2-\frac{1}{l_3^2}X_3^2+\frac{R}{6}+\mu^2\right)\phi=0 \tag{3}$$

where the X_i are the differential generators of the SU(2) (left) symmetry group.

The space–time is static, and previous work (Dowker and Kennedy 1978) shows that it is sufficient to discuss the spatial modes. Thus define the eigenvalues E^2 by

$$\left[-\frac{1}{l_1^2}\boldsymbol{X}^2-\left(\frac{1}{l_3^2}-\frac{1}{l_1^2}\right)X_3^2+\frac{R}{6}\right]\phi_{E^2}=E^2\phi_{E^2} \tag{4}$$

where $R=(2/l_1^2-l_3^2/2l_1^4)$.

The solutions of (4) are just the normalised rotation matrices

$$u^l_{mn}(q)=[l^{1/2}/(16\pi^2l_1^2l_3]^{1/2}]D^l_{mn}(q) \tag{5}$$

where $q\in\mathrm{SU(2)}$ and can be used to label the points of the spheroid because of the left SU(2) symmetry of the space. The index ranges are the usual ones, $l=2j+1=1,2,3,\ldots m,\ n=j,\ldots-j$ and E^2 depends on only l and m; $\boldsymbol{X}^2$ corresponds to J^2 and X_3 to J_z^{left}, up to a sign. The degeneracy of the eigenvalue E^2 is thus l (the range of n) and for $E^2_{l,m}$ we find

$$E^2_{l,m}=\frac{1}{l_1^2}\left[\frac{l^2}{4}+\alpha\left(m^2+\frac{\mathrm{i}}{12(1+\alpha)}\right)\right]+\mu^2 \tag{6}$$

where $1+\alpha=l_1^2/l_3^2$.

The Feynman Green function can be calculated in the standard way to give

$$G_4=\frac{\mathrm{i}}{2V}\sum_{J=0}^{\infty}(2J+1)\sum_{m=-J}^{J}\sum_{m=-J}^{J}\frac{\exp[-2E_{l,m}(t-t')]}{E_{l,m}}D^l_{mn}(\theta)D^l_{mn}(\theta')$$
$$\times\exp[\mathrm{i}m(\psi-\psi')]\exp[\mathrm{i}n(\phi-\phi')]. \tag{7}$$

The sum over J covers half-integral and integral values of J which correspond to a sum over l in equation (6).

The Green function satisfies

$$(\square+\mu^2+R/6)G_4=\delta(x,x'). \tag{8}$$

In order to compute the stress tensor we require a regularisation scheme, the most suitable one for the present case being the zeta function method (Dowker and Kennedy 1978). This entails finding a solution to

$$(\square+\mu^2+R/6)G_4^\nu=G_4^{\nu-1} \tag{9}$$

with the boundary condition that as $\nu\to1$ we regain equation (7).

The four-dimensional zeta function, G_4, and the zeta function on the spatial section, G_3, have the representations

$$G_4^\nu=\frac{\mathrm{i}^\nu}{\Gamma(\nu)}\int_0^\infty\frac{\mathrm{d}s}{s^{1-\nu}}\exp(-\mathrm{i}\mu^2s)\langle x'(s)|x''(0)\rangle_4,$$
$$G_3^\nu=\frac{\mathrm{i}^\nu}{\Gamma(\nu)}\int_0^\infty\frac{\mathrm{d}s}{s^{1-\nu}}\exp(-\mathrm{i}\mu^2s)\langle x'(s)|x''(0)\rangle_3, \tag{10}$$

where

$$\langle x'(s)|x''(0)\rangle_4 = (i/4\pi s)^{1/2} \exp[-i(t-t')^2/4s]\langle x'(s)|x''(0)\rangle_3$$

and is independent of ν.

The inverses of equations (10) are

$$\begin{aligned}\langle x'(s)|x''(0)\rangle_4 &= \frac{-i}{2\pi}(2s)^{1-\nu}\Gamma(\nu)\int_{-\infty}^{\infty} d\mu^2 \exp(i\mu^2 s)G_4^\nu,\\ \langle x'(s)|x''(0)\rangle_3 &= \frac{-i}{2\pi}(2s)^{1-\nu}\Gamma(\nu)\int_{-\infty}^{\infty} d\mu^2 \exp(i\mu^2 s)G_3^\nu.\end{aligned} \tag{11}$$

Since the $\langle x'(s)|x''(0)\rangle_4$ are independent of ν we have the freedom to put any value of ν in equation (11); in particular if we choose $\nu = 1$ we can substitute equation (11) into equation (10) and find an expression for G_4^ν in terms of G_4: the result is

$$G_4^\nu = \frac{(-1)^{\nu-1}}{\Gamma(\nu)}\left(\frac{d}{d\mu^2}\right)^{\nu-1} G_4 = \frac{(-1)^{\nu-1}}{\Gamma(\nu)}\left(\frac{d}{d(E^2)}\right)^{\nu-1} G_4 \tag{12}$$

which implies

$$G_4^\nu = \frac{(-1)^{\nu-1} i 2^{1-\nu}}{2\nu\Gamma(\nu)} \sum_{J=0}^{\infty} (2J+1) \sum_{-J}^{J}\sum_{-J}^{J} \frac{D^l_{mn}(\theta)D^l_{mn}(\theta)}{(E_{l,m})^{2\nu-1}} \exp[im(\psi-\psi^1)]\exp[in(\phi-\phi^1)]$$

$$\times \sum_{n=0}^{\infty} (-i)^n \frac{(t-t')^n}{n!} E^n_{J,K}(n-1)(n-3)(n-5)\ldots(n+3-2\nu). \tag{13}$$

Although (13) has been derived for integral ν, we can easily rewrite it in terms of gamma functions and thereby allow the ν to become arbitrary. It is interesting to note that the time dependence of equation (13) is only of exponential form when $\nu = 1$. It is easy to show that equation (13) satisfies equation (9) by direct substitution and use of equation (4). As far as $\langle T_\mu{}^\nu\rangle$ is concerned, we only need the expansion (13) up to $n = 2$.

3. Calculation of $\langle T_\mu{}^\nu\rangle$

It is conventional to write the vacuum energy–momentum tensor as the coincidence limit

$$\langle T_\mu{}^\nu\rangle = \lim_{x\to x'}\Big[-i\Big(\frac{2}{3}\partial_\mu\partial^{\nu'} - \frac{1}{6}g_\mu{}^\nu\,\partial_\sigma g^{\sigma\alpha}\partial_{\alpha'} - \frac{1}{3}\nabla_\mu\nabla^\nu$$

$$+\tfrac{1}{6}\mu^2 g_\mu{}^\nu + \tfrac{1}{36}g_\mu{}^\nu R\phi^2 - \tfrac{1}{6}R_\mu{}^\nu\Big)G_4^\nu(x,x')\Big]. \tag{14}$$

A peculiarity of the zeta function method is that the operator $\Box + \mu^2 + R/6$ in equation (14) produces, by virtue of equation (9), a finite quantity. In other regularisation schemes this is not so, and one usually uses equation (4) to remove the corresponding term in equation (14). So is there a possibility of an ambiguity being present in our regularisation scheme? To take this possibility into account we add to the right-hand side of equation (14) the term

$$-\tfrac{1}{2}\mathrm{i}(\lambda-1)g_\mu{}^\nu(\Box+\mu^2+R/6)G_4^\nu.$$

Our reason for doing this will become apparent shortly.

We can simplify equation (14) for the case in hand by noting that the properties of the d^J_{km} imply (Vilenkin 1968)

$$\frac{\partial}{\partial\theta}G_4^\nu|_{\theta\to\theta'}\to 0, \qquad \frac{\partial^2}{\partial\theta\partial\theta'}G_4^\nu|_{\theta\to\theta'}=\frac{-\partial^2}{\partial\theta^2}G_4^\nu|_{\theta\to\theta'} \tag{15}$$

so that

$$\phi_{,\sigma}g^{\sigma\alpha}\phi_{,\alpha}\sim-\phi\Box\phi$$

and

$$\partial_\mu\partial_{\nu'}\to-\partial_\mu\partial_\nu, \qquad \nabla_\mu\nabla^\nu\to\partial_\mu\partial^\nu$$

and equation (14) becomes

$$\langle T_\mu{}^\nu\rangle=-\mathrm{i}[-g^{\nu s}\,\partial_\mu\partial_s-\tfrac{1}{6}R_\mu{}^\nu+\tfrac{1}{2}\lambda g_\mu{}^\nu(\Box+\mu^2+\tfrac{1}{6}R)]G_4^\nu. \tag{16}$$

We will show that only for the situation $\lambda=0$ is there a consistent renormalisation possible for the fundamental constants of the system (e.g. Newton's constant) when the zeta function method is invoked.

If we wanted, we could avoid this problem entirely by returning to the effective Lagrangian, renormalising and then obtaining the renormalised $\langle T_\mu{}^\nu\rangle$ by variation. This is the only foolproof method. (In fact, because of the symmetries of the manifold, one can use this method to avoid local expressions like equation (14) altogether, as we shall see later.)

In order to demonstrate the consistency of our renormalisation scheme, we first note that the effective Lagrangian is given by

$$\mathscr{L}^\nu=[-\mathrm{i}/2(\nu-1)]G_4^{\nu-1}. \tag{17}$$

If we substitute the expansion

$$G^\nu=\frac{\mathrm{i}}{16\pi^2}\left(\frac{(\mu^2)^{2-\nu}}{(\nu-1)(\nu-2)}+\frac{a_1(\mu^2)^{1-\nu}}{\nu-1}+a_2(\mu^2)^{-\nu}+\ldots\right) \tag{18}$$

(which is derivable from equation (10), using DeWitt's (1965) expansion for the $\langle x'(s)|x''(0)\rangle$ into equation (10), then we can derive an expansion for the effective Lagrangian in terms of the a_n. By adding the classical Lagrangian to this expansion, we can obtain values for the renormalised Newton's constant in terms of the bare values and a curvature-dependent addend. We can clearly do something similar for the energy-momentum tensor via equation (15), if we substitute equation (15) into the right-hand side of the Einstein equations

$$-\langle T_\mu{}^\nu\rangle=g_\mu{}^\nu\Lambda+(1/8\pi G)(R_\mu{}^\nu-\tfrac{1}{2}g_\mu{}^\nu R).$$

The conclusions are

$$\Lambda_{\text{ren}}=\Lambda-\frac{1}{32\pi^2}\frac{(\mu^2)^{3-\nu}}{(\nu-3)(\nu-2)(\nu-1)}, \qquad \frac{R}{16\pi G_{\text{ren}}}=\frac{R}{16\pi G}+\frac{a_1(\mu^2)^{2-\nu}}{32\pi^2(\nu-1)(\nu-2)}, \tag{19}$$

from equation (17) and

$$\Lambda_{\text{ren}} = \Lambda + \frac{1}{32\pi^2}[\lambda(\nu-1)-1]\frac{(\mu^2)^{3-\nu}}{(\nu-1)(\nu-2)(\nu-3)}, \tag{19}$$

$$\frac{R}{16\pi G_{\text{ren}}} = \frac{R}{16\pi G} + [1-2\lambda(\nu-1)]\frac{a_1(\mu^2)^{2-\nu}}{(\nu-1)(\nu-2)}, \tag{20}$$

from equation (16).

Thus, if we use the zeta function method, self-consistency arguments force us to drop the third term on the right-hand side of equation (16). In what follows we shall assume that this has been done. This is simply a technical device that produces the correct coincidence limit expression for $\langle T_\mu{}^\nu\rangle$.

It is of interest to note that the trace of $\langle T_\mu{}^\nu\rangle$ satisfies

$$\langle T_\mu{}^\nu\rangle = -\mathrm{i}(\mu^2 G_4^\nu - G_4^{\nu-1}). \tag{21}$$

If one uses equation (18), then it is clear that equation (21) contains no term which is quadratic in the curvature, so that as far as the massless case is concerned, no renormalisation need be done in computing $\langle T_\mu{}^\nu\rangle$, and for $m=0$ equation (21) is the statement of the trace anomaly.

On performing the differentiations in equations (16), we find that the $\langle T_\mu{}^\nu\rangle$ can be expressed in terms of three independent functions,

$$2V\langle T_0{}^0\rangle = \left[\mu^2 a(\nu) + \frac{1}{3a^2}\alpha(1-\alpha)a(\nu) + \frac{b(\nu)}{a^2} + \frac{4\alpha(\nu)}{a^2}\right] - a(\nu-1),$$

$$2V\langle T_1{}^1\rangle = (1/6a^2)(1-2\alpha+2\alpha^2)a(\nu) - (1/2a^2)(b(\nu)-4c(\nu)) = \langle T_2{}^2\rangle 2V,$$

$$2V\langle T_3{}^3\rangle = -(4/a^2)(1+\alpha)c(\nu) - (1/3a^2)(1-\alpha+\alpha^2)a(\nu),$$

$$2V\langle T_2{}^3\rangle = \frac{\cos\theta}{2a^2}(b(\nu)-a(\nu)-12c(\nu)) - \frac{4\alpha c(\nu)}{a^2}\cos\theta + \frac{2\alpha}{3a^2}(1-\alpha)a(\nu)\cos\theta, \tag{22}$$

where

$$a(\nu) = \frac{\Gamma(\nu-\frac{1}{2})}{\Gamma(\nu)\sqrt{\pi}}\sum_{J=0}^{\infty}(2J+1)\sum_{-J}^{J}(E_{l,m})^{1-2\nu}$$

$$c(\nu) = \frac{\Gamma(\nu-\frac{1}{2})}{\Gamma(\nu)\sqrt{\pi}}\sum_{J=0}^{\infty}(2J+1)\sum_{-J}^{J}m^2 E_{l,m}^{1-2\nu},$$

$$b(\nu) = \frac{\Gamma(\nu-\frac{1}{2})}{\Gamma(\nu)\sqrt{\pi}}\sum_{J=0}^{\infty}(2J+1)^3\sum_{-J}^{J}(E_{l,m})^{1-2\nu},$$

$$G_4^\nu(x,x) = \frac{\mathrm{i}}{2V}a(\nu) \tag{23}$$

(we take $\nu\to 1$ for the physical limit).

To evaluate the functions we expand E_{lm} in terms of small α, up to order α^2. In doing this we encounter summations such as

$$\phi_{-\nu} = \sum_{n=1}^{\infty}(n^2+\mu^2 a^2)^{-\nu} = \sum_{J=0}^{\infty}[(J+\tfrac{1}{2})^2+\mu^2 l_1^2]^{-\nu}2^{-2\nu}, \tag{24}$$

to evaluate these, we write

$$\phi_{-\nu}=\frac{1}{\Gamma(\nu)}\sum_{n=1}^{\infty}\int_{n=1}^{\infty}\frac{dx}{x^{1-\nu}}\exp[-x(n^2+\mu^2a^2)]. \tag{25}$$

The summation can be performed using (Gradshteyn and Ryzhik 1980)

$$\frac{1}{2\Gamma(\nu)}\int_0^{\infty}\frac{dx}{x^{1-\nu}}\exp(-x\mu^2a^2)[v_3(0|2x/\pi)-1]$$

$$=\frac{1}{\Gamma(\nu)}\int_0^{\infty}\frac{dx}{x^{1-\nu}}\exp[-x\mu^2a^2]$$

$$\times\left(-\frac{1}{2}+\frac{1}{2(x/\pi)^{1/2}}+\frac{1}{2(x/\pi)^{1/2}}[\theta_3(0|i\pi/x)-1]\right). \tag{26}$$

Finally, if we use the representation

$$\theta_3(0|i\pi/x)-1=\sum_{n=1}^{\infty}\exp(-n^2\pi^2/x)$$

and perform the integration, we find

$$\phi_{-\nu}=\frac{-1}{2(\mu^2a^2)^{\nu}}+\frac{\sqrt{\pi}\Gamma(\nu-\frac{1}{2})}{2(\mu^2a^2)^{\nu-1/2}\Gamma(\nu)}+\frac{2\sqrt{\pi}}{\Gamma(\nu)}\sum_{n=1}^{\infty}\frac{K_{\nu-\frac{1}{2}}(2n\pi\mu a)}{(\mu a/n\pi)^{\nu-1/2}}. \tag{27}$$

As a check it is possible to take the massless limit of equation (27) and one obtains $\phi_{-\nu}=\xi(2\nu)$, as should be obvious from equation (24).

Since the form that the functions a, b, and c take in terms of the $\phi_{-\nu}$ is rather long, we relegate these terms to the Appendix.

As far as our regularisation scheme for the massive case is concerned, it turns out that the class of terms which derive from the second term in equation (27) leads to a renormalisation of the cosmological constant etc. The first term in equation (27) is a red herring in the sense that eventually it is cancelled, so that finally all we have is a series of Bessel functions of integral order. The renormalised stress tensor can then be derived from equations (22) and (A1.1) but with $\nu\to 1$.

It should be clear from the above discussion that we are going to encounter difficulties in taking the massless limit of our renormalised stress tensor. In the massless limit both the second and third terms in equation (27) have $\ln m^2$ behaviour (such that they both cancel each other). Throwing away the second term, therefore, will produce an infrared divergence in $\langle T_\mu{}^\nu\rangle_{\text{ren}}$ and in $\mathcal{L}_{\text{ren}}$. We thus cannot obtain the massless limit of $\langle T_\mu{}^\nu\rangle_{\text{ren}}$, although we can obtain the trace anomaly by $\langle T_\mu{}^\nu\rangle_{\text{ren}}=-2m^2\,\partial/\partial m^2\mathcal{L}_{\text{ren}}$.

A similar situation has been encountered by Bunch and Davies (1978). These authors consider a non-conformally coupled scalar field in a Robertson–Walker space–time. Their renormalisation prescription is to subtract from the massless Green function a DeWitt–Schwinger expansion up to order R^2. This brings in an infrared divergence of the type mentioned above in their expressions for the stress tensor. Bunch and Davies have shown in the massless limit that these infrared divergences can be absorbed into the renormalisation of the coupling constants associated with terms quadratic in the curvature in the classical Lagrangian. After removal of these divergences Bunch and Davies then take the massless limit.

Fortunately, for the case of the squashed Einstein universe we can examine the rigorously massless scalar field without introducing any spurious infrared divergences, and we will now show how this can be done.

4. The massless limit

The massless limit of equation (22) is

$$\langle T_0{}^0\rangle = -\mathscr{L} = -\frac{\alpha^2}{180\pi^2 a^4}\left(\frac{1}{\nu-1}+2-2\ln 2+2\gamma+\ln\frac{a^2}{L^2}\right) + \frac{1}{480\pi^2 a^4}\left(1+\frac{2}{3}\alpha+\alpha^2\frac{47}{45}\right)+\mathrm{O}(\nu-1). \tag{28}$$

The other $\langle T_i^j\rangle$ are given by equations (23) but with

$$\begin{aligned}\frac{1}{a}c(\nu) &= \frac{1}{120}\left(\frac{11}{12}-\frac{259}{360}\alpha+\frac{3023}{2016}\alpha^2-\frac{71}{630}\alpha^2\zeta(3)\right)+\left(-\frac{\alpha}{45}+\frac{\alpha^2}{12}\right)\Omega+\frac{13\alpha^2}{216},\\ \frac{1}{a}b(\nu) &= \frac{1}{120}\left(1-\frac{\alpha}{6}+\frac{3\alpha^2}{40}\right)+\frac{\alpha^2}{15}\Omega+\frac{2\alpha^2}{45},\\ \frac{1}{a}a(\nu) &= -\frac{1}{12}\left(1-\frac{\alpha}{6}+\frac{3\alpha^2}{40}\right)+\frac{2\alpha^2}{15}\zeta(3)-\frac{\alpha^2}{18},\\ \Omega &= 1/(\nu-1)+2-2\ln 2+2\gamma+\ln(a^2/L^2)\equiv\bar{\Omega}+\ln(a^2/L^2).\end{aligned} \tag{29}$$

Our regularisation scheme is to subtract a term from $\mathscr{L}$ proportional to $R^2-3R_{\mu\nu}R^{\mu\nu}$ and correspondingly a term from $\langle T_{\mu\nu}\rangle$ proportional to $(2/\sqrt{-g})(\delta/\delta g_{\mu\nu})\sqrt{-g}(R^2-3R_{\mu\nu}R^{\mu\nu})$, the constant of proportionality being chosen so that we can remove the $\bar{\Omega}$ terms which contain the poles. Our final result for $\langle T_{\mu\nu}\rangle_{\text{ren}}$ is found to be merely equation (29) without the $\bar{\Omega}$ terms. The same value can also be obtained from the massive case if we adopt Bunch and Davies' arguments for removing the infrared divergences.

We note in passing that there are two ways to compute the variation of the curvature and metric-dependent terms in the Lagrangian. We can either substitute the appropriate squashed Einstein values into the right-hand side of

$$\frac{1}{\sqrt{-g}}\frac{\delta}{\delta g_{\mu\nu}}\int \mathrm{d}^4x\,\sqrt{-g}R_{\mu\nu}R^{\mu\nu} = g^{\mu\nu}R^{\sigma\tau}{}_{;\sigma\tau}+\Box R^{\mu\nu}-R^{\mu\sigma}{}_{;}{}^{\nu}{}_{\sigma}-R^{\nu\sigma}{}_{;}{}^{\mu}{}_{\sigma} -2R^{\mu}{}_{\sigma}R^{\nu\sigma}+\tfrac{1}{2}g^{\mu\nu}R_{\sigma\tau}R^{\sigma\tau}, \tag{30}$$

which is tedious, or more simply we can rewrite $\delta/\delta g_{\mu\nu}$ in terms of $\delta/\delta a$ and $\delta/\delta\alpha$ and vary equation (28) directly. Both methods are found to be in agreement. In particular, for the latter method we can write

$$\begin{aligned}l_3^2\,\partial/\partial l_3^2 &= l_3^2\,(\partial/\partial l_3^2)g_{ij}\,\partial/\partial g_{ij}\\ &= g'_{22}\,\partial/\partial g_{22}+g'_{33}\,\partial/\partial g_{33}+2g'_{32}\,\partial/\partial g_{32}\end{aligned}$$

where the prime denotes $l_3^2\ \partial/\partial l_3^2$; hence

$$-(2/\sqrt{-g})l_3^2\ (\partial/\partial l_3^2)\sqrt{-g}\mathscr{L}$$
$$=-(2/\sqrt{-g})[g'_{22}(T_2{}^2g^{22}+T_3{}^2g^{32})+g'_{33}(T_3{}^3g^{33}+T_2{}^3g^{23})$$
$$+2g'_{32}(T_3{}^2g^{33}+T_2{}^2g^{23})]$$

but

$$T_3{}^2=0, \qquad T_2{}^3=(T_3{}^2-T_2{}^2)\cos\theta$$

imply

$$-\frac{2}{\sqrt{-g}}l_3{}^2\frac{\partial}{\partial l_3{}^2}\sqrt{-g}\,\mathscr{L}=T_3{}^3=-\frac{2}{\sqrt{-g}}g_{33}\frac{\delta}{\delta g_{33}}\sqrt{-g}\mathscr{L};$$

similarly

$$-(1/\sqrt{-g})l_1^2(\partial/\partial l_1^2)\sqrt{-g}\mathscr{L}=T_1{}^1;$$

clearly, since

$$\frac{\partial}{\partial\alpha}=\frac{\partial l_3^2}{\partial\alpha}\frac{\partial}{\partial l_3^2}=\frac{(1+2\alpha-3\alpha^2)}{(1-\alpha+\alpha^2)}l_3{}^2\frac{\partial}{\partial l_3^2}$$

and

$$a^2\ \partial/\partial a^2=l_1^2\ \partial/\partial l_1^2+l_3^2\ \partial/\partial l_3^2,$$

we find

$$\langle T_1{}^1\rangle=-\frac{1}{\sqrt{-g}}\left(a^2\frac{\partial}{\partial a^2}+\frac{1-\alpha+\alpha^2}{1-2\alpha+3\alpha^2}\frac{\partial}{\partial\alpha}\right)\sqrt{-g}\mathscr{L}. \tag{31}$$

The pole term in $\mathscr{L}$ (equation (28)) implies that $T_1{}^1$ has a pole term given by

$$\langle T_1{}^1\rangle=(-180\pi^2a^4)^{-1}(2\alpha-5\alpha^2)(\nu-1)^{-1}. \tag{32}$$

This value agrees with equations (29) and (23) and our regularisation scheme is valid.

By varying equation (28) we can also determine the anomaly from

$$\langle T_\mu{}^\mu\rangle=\langle T_0{}^0\rangle-(2/\sqrt{-g})(l_1^2\ \partial/\partial l_1^2+l_3^2\ \partial/\partial l_3^2)\sqrt{-g}\mathscr{L} \tag{33}$$
$$=-\mathscr{L}-(2/\sqrt{-g})a^2\ (\partial/\partial a^2)\sqrt{-g}\mathscr{L}=-\alpha^2/90\pi^2a^4.$$

That this is non-zero is due to the presence of the $\ln(a^2/L^2)$ in the Lagrangian.

5. Alternative approach

From the foregoing discussion it should be clear that the Lagrangian alone is sufficient to determine all the $T_\mu{}^\nu$, although we would need to find it to order α^3 if we use the above variational method.

In this alternative approach, because of the static nature of the metric, all one needs is the integrated zeta function $\zeta_3(\nu)$ on the spatial section given by

$$\zeta_3(\nu)=l_1^{2\nu}\sum_{l,m}l\left[\frac{l^2}{4}+\alpha\left(m^2+\frac{1}{12(1+\alpha)}\right)\right]^{-\nu} \tag{34}$$

in the massless case, for simplicity. This, when expanded in powers of α, gives

$$\zeta_3(\nu)=a^{2\nu}\{\zeta_R(2\nu-2)-\tfrac{1}{3}\alpha\nu\zeta_R(2\nu-2)+4\alpha^2(\tfrac{1}{12}\nu\zeta_R(2\nu)+2\nu(\nu+1)\times[\tfrac{1}{80}\zeta_R(2\nu-2)-\tfrac{1}{36}\zeta_R(2\nu)+\tfrac{1}{45}\zeta_R(2\nu+2)]\}+O(\alpha^3) \quad (35)$$

in terms of the Riemann zeta function ζ_R.

The effective Lagrangian is given as the spatial integral of equation (17) in terms of the equal-time four-dimensional zeta function

$$\int \mathscr{L}\sqrt{-g}\,d^3x=\frac{-i}{2}\lim_{\nu\to 1}\frac{\zeta_4(\nu-1)}{\nu-1}L^{-2\nu+2}$$
$$=\frac{-i}{2}\lim_{\nu\to 1}\left[\zeta_4(0)\left(\frac{1}{\nu-1}+\ln L^2\right)+\zeta_4'(0)\right]. \quad (36)$$

For static space–times $\zeta_4(\nu)$ can be related (Dowker and Kennedy 1978) to $\zeta_3(\nu)$:

$$\zeta_4(\nu)=\frac{i}{(4\pi)^{1/2}}\frac{\Gamma(\nu-\frac{1}{2})}{\Gamma(\nu)}\zeta_3(\nu-\tfrac{1}{2}). \quad (37)$$

We shall not pursue this method much further, but shall just point out that $\zeta_4(0)$ is non-zero here, in contrast to the spherical Einstein case. This is easily seen because the above explicit form for $\zeta_3(\nu)$ has a pole at $\nu=-\frac{1}{2}$, coming from the final $\zeta_R(2\nu+2)$.

The general value of $\zeta_4(0)$ is given in terms of the proper-time expansion coefficient a_2 as (Dowker and Critchley 1977)

$$\zeta_4(0)=\frac{i}{16\pi^2}\int a_2(-g)^{1/2}\,d^3x. \quad (38)$$

The expression for a_2 is standard and is

$$a_2=\tfrac{1}{180}(R_{\alpha\beta\gamma\delta}R^{\alpha\beta\gamma\delta}-R_{\alpha\beta}R^{\alpha\beta}). \quad (39)$$

To order α^2 we find $a_2\sim 8\alpha^2/45a^4$ and so $\zeta_4(0)\sim i\alpha^2/45a$.

The reader can check that this agrees with the explicit pole in $\zeta_3(\nu)$. It also gives the trace anomaly correctly.

6. Proof that the $T_\mu{}^\nu$ can self consistently sustain a squashed Einstein universe

We shall show that for certain values of the radius a, and the distortion parameter, that the renormalised energy–momentum tensor for the massless scalar field can satisfy Einstein's equations i.e.

$$-\langle T_{\mu\nu}\rangle_{\rm ren}=\Lambda_{\rm ren}g_{\mu\nu}+(1/8\pi G_{\rm ren})(R_{\mu\nu}-\tfrac{1}{2}g_{\mu\nu}R). \quad (40)$$

Defining the trace of $\langle T_\mu{}^\nu\rangle$ by T and expanding both sides of equation (40) in terms of T, we find $(8\pi G=1)$

$$\begin{aligned}
&\langle T_0{}^0\rangle-\tfrac{1}{4}T=(3+\alpha-\alpha^2)/2a^2=\bar{A}_1+\bar{B}_1\alpha+\bar{C}_1\alpha^2,\\
&\langle T_2{}^2\rangle-\tfrac{1}{4}T=\langle T_1{}^1\rangle-\tfrac{1}{4}T=-(1+3\alpha+3\alpha^2)/2a^2=\bar{A}_2+\bar{B}_2\alpha+\bar{C}_2\alpha^2,\\
&\langle T_3{}^3\rangle-\tfrac{1}{4}T=-(1/2a^2)(1-5\alpha+5\alpha^2)=\bar{A}_3+\bar{B}_3\alpha+\bar{C}_3\alpha^2,\\
&\langle T_2{}^3\rangle=(4\alpha/a^2)\cos\theta(1-\alpha)=\cos\theta(\bar{B}_4\alpha+\bar{C}_4\alpha^2).
\end{aligned} \quad (41)$$

Luckily the coefficients on the right-hand side, i.e. the $\bar{A}$, $\bar{B}$ and $\bar{C}$, are not independent; for example, we can invoke the tracelessness of equations (28) and (29) to show

$$\bar{A}_3+2\bar{A}_2+\bar{A}_1=0, \qquad \bar{B}_3+2\bar{B}_2+\bar{B}_1=0, \qquad \bar{C}_3+2\bar{C}_2+\bar{C}_1=0.$$

Finally the divergenceless condition $\nabla_\nu\langle T_\mu{}^\nu\rangle=0$ can be used to show $\bar{A}_2=\bar{A}_3$, $\bar{B}_3=\bar{B}_2+\bar{B}_4$, $\bar{C}_3=\bar{C}_2+\bar{C}_4$ which corresponds to $T_2{}^3=\cos\theta\,(T_3{}^3-T_1{}^1)$. Thus eleven *a priori* independent coefficients have been reduced by these six conditions to give just five independent coefficients. If we choose $\bar{B}_4$, $\bar{C}_4$, $\bar{A}_1$, $\bar{B}_1$ and $\bar{C}_1$ as our independent coefficients, then from equation (41)

$$\begin{aligned} 3/2a^2&=\bar{A}_1+(B_1-\tfrac{1}{8}\bar{B}_4)\alpha+(\bar{C}_1-\tfrac{1}{8}\bar{C}_4)\alpha^2+\ldots,\\ 0&=(4/a^2-\bar{B}_4)\alpha-(\bar{C}_4+4/a^2)\alpha^2+\ldots, \end{aligned} \tag{42}$$

i.e. two equations for two unknowns, α, a. Solutions are possible to all orders in α, although we stop at order α^2.

In equation (34),

$$\begin{aligned} &\bar{A}_1=\frac{1}{480\pi^2a^4}, \qquad \bar{B}_1=\frac{1}{720\pi^2a^4}, \qquad \bar{C}_1=\frac{107}{21600\pi^2a^4}-\frac{\ln(a^2/L^2)}{180\pi^2a^4},\\ &\bar{B}_4=-39/2700\pi^2a^4+(1/30\pi^2a^4)\ln(a^2/L^2),\\ &\bar{C}_4=-\frac{199\,583}{2419\,200\pi^2a^4}-\frac{769\zeta(3)}{50\,400\pi^2a^4}-\frac{7}{90\pi^2a^4}\ln a^2/L^2. \end{aligned} \tag{43}$$

Solutions are possible which satisfy $\alpha \ll 1$, but only for a certain range of values of L/a.

In order to obtain some sort of feel for the numbers involved here, we will assume certain values of the ratios L/a, substitute these into equations (42) and solve for α and a. We will then see if our $\alpha \ll 1$ condition holds.

From equation (42) (to order α only, for simplicity)

$$\frac{3}{2a^2}=\bar{A}_1+\left(\bar{B}_1-\frac{\bar{B}_4}{8}\right)\frac{(4/a^2-\bar{B}_4)}{(4/a^2+\bar{C}_4)}. \tag{44}$$

Equation (44) is then easily solved for a^2. Rewriting $A_1=f/a^4$, $\bar{B}_1-\frac{1}{8}\bar{B}_4=d/a^4$, $B_4=4b/a^4$ and $C_4=4c/a^4$, we find

$$\begin{aligned} a^2&=-\frac{h\pm(h^2-12j)^{1/2}}{6}, \qquad h=3c-2f-2d,\\ \alpha&=\frac{1-b/a^2}{1+c/a^2}, \qquad j=2db-2fc. \end{aligned} \tag{45}$$

If we choose the values for $L/a \gg 1$, $L/a \ll 1$, $L/a=1$ and $L/a=0.7$ then we find, respectively, $a=1.1\times10^{-2}(\ln L^2/a^2)^{1/2}$, $4.3\times10^{-2}(\ln a^2/L^2)^{1/2}$, 1×10^{-2}, 1.1×10^{-2} and $\alpha=0.43$, -0.43, -0.19, 0.027. If we wish to reinstate the units we should multiply all quantities which have the dimensions of length by $(8\pi\hbar G/c^3)^{1/2}$.

7. Conclusions

These results show that a squashed Einstein universe can self consistently be maintained by the vacuum energy–momentum tensor of a massless scalar field. One would expect the same of other fields. Of course the most interesting question is the stability of this self-supporting solution.

It is a classical result, due to deSitter, that the spherical Einstein universe is unstable against radial perturbations, and one would like to establish a corresponding result in the self-consistent case for both radius and shape deformations. This we defer to another time.

A final qualification: since we have totally ignored graviton problems, we do not claim that the self-consistent set-up of § 6 has any connection with reality nor that it is technically consistent, in a field-theoretic sense.

In checking our approximation $\alpha \ll 1$, we found that, although it was a reasonable approximation for $L/a \sim 1$, it was unreliable for other values for L/a, as the results quoted above suggest.

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Appendix 1.

In this appendix we list the values of some functions defined earlier in the text. They are particularly nasty expressions because they have been calculated for the case of a massive scalar field. The massless limit of these functions has already been quoted in the main body of this paper.

$$a(\nu)=\frac{\Gamma(\nu-\frac{1}{2})}{\Gamma(\nu)\sqrt{\pi}}[A(\nu)\phi_{3/2-\nu}+B(\nu)\phi_{1/2-\nu}+C(\nu)\phi_{-1/2-\nu}+D(\nu)\phi_{-3/2-\nu},$$

$$A(\nu)=a^{2\nu-1}[1-\tfrac{1}{3}\alpha(\nu-\tfrac{1}{2})+\tfrac{1}{10}\alpha^2(\nu^2-\tfrac{1}{4})],$$

$$B(\nu)=a^{2\nu-1}[\mu^2a^2(-1+\tfrac{2}{3}\alpha(\nu-\tfrac{1}{2})-\tfrac{3}{10}\alpha^2(\nu^2-\tfrac{1}{4}))+\tfrac{2}{9}\alpha^2(\nu-\tfrac{1}{2})(1-\nu)],$$

$$C(\nu)=a^{2\nu-1}[\mu^4a^4(-\tfrac{1}{3}\alpha(\nu-\tfrac{1}{2})+\tfrac{3}{10}\alpha^2(\nu^2-\tfrac{1}{4}))$$
$$+\mu^2a^2(\tfrac{4}{9}\alpha^2(\nu^2-\tfrac{1}{4})-\tfrac{1}{3}\alpha^2(\nu^2-\tfrac{1}{2}))+\tfrac{8}{45}\alpha^2(\nu^2-\tfrac{1}{4})],$$

$$D(\nu)=-8\alpha^2a^{2\nu-1}(\nu^2-\tfrac{1}{4})[\mu^2a^2/45+\mu^4a^4/36+\mu^6a^6/80],$$

$$b(\nu)=(-\mu^2a^2a(\nu)+[A(\nu)\phi_{5/2-\nu}+B(\nu)\phi_{3/2-\nu}$$
$$+C(\nu)\phi_{1/2-\nu}+D(\nu)\phi_{-1/2-\nu}])\Gamma(\nu-\tfrac{1}{2})/\sqrt{\pi}\Gamma(\nu),$$

$$C(\nu)=\frac{\Gamma(\nu-\frac{1}{2})}{\sqrt{\pi}\Gamma(\nu)}[E(\nu)\phi_{5/2-\nu}+F(\nu)\phi_{3/2-\nu}+G(\nu)\phi_{1/2-\nu}$$
$$+H(\nu)\phi_{-1/2-\nu}+L(\nu)\phi_{-3/2-\nu}],$$

$$
\begin{aligned}
E(\nu) &= a^{2\nu-1}[1\tfrac{1}{2}-\tfrac{1}{20}\alpha(\nu-\tfrac{1}{2})+\tfrac{1}{56}\alpha^2(\nu^2-\tfrac{1}{4})],\\
F(\nu) &= a^{2\nu-1}[(-1\tfrac{1}{2}-\tfrac{1}{6}\mu^2a^2)+\alpha(\nu-\tfrac{1}{2})(\tfrac{3}{20}\mu^2a^2+\tfrac{5}{36})\\
&\quad +\alpha^2(\nu-\tfrac{1}{2})(\tfrac{1}{36}-\tfrac{1}{14}\mu^2a^2(\nu+\tfrac{1}{2})-\tfrac{13}{120}(\nu+\tfrac{1}{2}))],\\
G(\nu) &= a^{2\nu-1}\{\tfrac{1}{12}\mu^4a^2+\tfrac{1}{12}\mu^2a^2)-\alpha(\nu-\tfrac{1}{2})(\tfrac{4}{45}+\tfrac{5}{18}\mu^2a^2+\tfrac{3}{20}\mu^4a^4)\\
&\quad +\alpha^2(\nu-\tfrac{1}{2})[-\tfrac{1}{18}\mu^2a^2-\tfrac{1}{36}+(\nu+\tfrac{1}{2})(\tfrac{3}{28}\mu^4a^4+\tfrac{13}{40}\mu^2a^2+\tfrac{13}{54})]\},\\
H(\nu) &= a^{2\nu-1}\{\alpha(\nu-\tfrac{1}{2})(\tfrac{1}{20}\mu^6a^6+\tfrac{5}{36}\mu^4a^4+\tfrac{4}{45}\mu^2a^2)+\alpha^2(\nu-\tfrac{1}{2})\\
&\quad \times[\tfrac{1}{36}\mu^4a^4+\tfrac{1}{36}\mu^2a^2+(\nu+\tfrac{1}{2})(-\tfrac{1}{14}\mu^6a^6-\tfrac{13}{40}\mu^4a^4-\tfrac{13}{28}\mu^2a^2-\tfrac{142}{945})]\},\\
L(\nu) &= a^{2\nu-1}[\alpha^2(\nu-\tfrac{1}{2})(\mu^8a^8+\tfrac{13}{120}\mu^6a^6+\tfrac{13}{54}\mu^4a^4+\tfrac{142}{945}\mu^2a^2)],
\end{aligned}
$$

$$
\begin{aligned}
&\phi_{-\nu} = -\frac{1}{2(\mu^2a^2)^\nu}+\frac{\sqrt{\pi}\Gamma(\nu-\frac{1}{2})}{2(\mu^2a^2)^{\nu-1/2}\Gamma(\nu)}+\phi_{-\nu},\\
&\psi_{-\nu} = \frac{2\sqrt{\pi}}{\Gamma(\nu)}\sum_{n=1}^{\infty}\frac{K_{\nu-1/2}(-2n\pi\mu a)}{(\mu a/n\pi)^{\nu-1/2}},\\
&4\pi^2a^2\bar{A}_n = A_n, \qquad 4\pi^2a^2\bar{B}_n = B_n+\tfrac{1}{2}A_n,\\
&4\pi^2a^2\bar{C}_n = C_n+B_n-\tfrac{1}{8}A_n, \qquad n=1,2,3,\\
&4\pi^2a^3\bar{B}_4 = B_4, \qquad 4\pi^2a^3\bar{C}_4 = C_4+\tfrac{1}{2}B_4,
\end{aligned}
\tag{A1.1}
$$

$$
\begin{aligned}
A_1 &= -\tfrac{5}{4}\mu^2\psi_{1/2}+a^{-2}\psi_{3/2}+\tfrac{1}{4}\mu^4a^2\psi_{-1/2},\\
B_1 &= -\tfrac{7}{24}\mu^2\psi_{1/2}+\tfrac{1}{12}\mu^4a^2\psi_{-1/2}+\tfrac{1}{24}\psi_{-3/2}\mu^6a^4+\psi_{3/2}(6a^2)^{-1},\\
C_1 &= \psi_{1/2}(\tfrac{9}{160}\mu^2(1/9a^2))+\psi_{-1/2}(-\tfrac{3}{160}\mu^4a^2+\tfrac{1}{18}\mu^2-\tfrac{2}{45})\\
&\quad +\psi_{-3/2}(-\tfrac{1}{32}\mu^6a^4+\tfrac{1}{72}\mu^4a^2+\tfrac{1}{90}\mu^2)\\
&\quad +\psi_{-5/2}(-\tfrac{9}{160}\mu^8a^6-\tfrac{9}{72}\mu^6a^4-\tfrac{1}{10}\mu^4a^2)-\tfrac{1}{40}a^{-2}\psi_{3/2},\\
B_4 &= -\tfrac{4}{15}a^{-2}\psi_{3/2}+\frac{1}{a^2}(\tfrac{7}{15}\mu^2a^2+\tfrac{2}{3})\psi_{1/2}+a^{-2}\psi_{-1/2}\\
&\quad \times(-\tfrac{2}{15}\mu^4a^2-\tfrac{1}{3}\mu^2a^2+\tfrac{4}{15})+a^{-2}\psi_{-3/2}(-\tfrac{1}{15}\mu^6u^6-\tfrac{1}{3}\mu^4a^4-\tfrac{4}{15}\mu^2),\\
C_4 &= \psi_{3/2}\tfrac{26}{455}a^{-2}+a^{-2}\psi_{1/2}(-\tfrac{31}{45}-\mu^2a^2\tfrac{9}{70})+a^{-2}\psi_{-1/2}(\mu^4a^4\tfrac{3}{70}+\mu^2a^2\tfrac{31}{90}-\tfrac{34}{45})\\
&\quad +a^{-2}\psi_{-3/2}(\tfrac{1}{14}\mu^6a^6+\mu^4a^4\tfrac{32}{45}+\mu^2a^2\tfrac{4061}{2520}+\tfrac{192}{315})\\
&\quad +a^{-2}\psi_{-5/2}(-\tfrac{69}{560}\mu^8a^8-\tfrac{11}{30}\mu^6a^6-\tfrac{742}{795}\mu^4a^4-\tfrac{64}{105}\mu^2a^2).
\end{aligned}
\tag{A1.2}
$$

Appendix 2. Computation of the Riemann tensor

The metric of the space–time is given by

$$
\begin{aligned}
&g_{00}=1, \qquad g_{11}=-l_1^2, \qquad g_{22}=-(l_1^2\sin^2\theta+l_3^2\cos^2\theta),\\
&g_{33}=-l_3^2, \qquad g_{32}=-l_3^2\cos\theta=g_{23}.
\end{aligned}
\tag{A2.1}
$$

Its inverse is

$$
\begin{aligned}
&g^{00}=1, \qquad g^{11}=-1/l_1^2, \qquad g^{22}=-1/l_1^2\sin^2\theta,\\
&g^{33}=-1/l_3^2-(\cot^2\theta)/l_1^2, \qquad g^{23}=g^{32}=(\cos\theta)/l_1^2\sin^2\theta.
\end{aligned}
\tag{A2.2}
$$

The Christoffel symbols are defined by

$$\Gamma^{s}_{\mu\nu} = \tfrac{1}{2} g^{s\alpha}(g_{\alpha\mu,\nu} + g_{\alpha\nu,\mu} - g_{\mu\nu,\alpha}) \qquad \text{(A2.3)}$$

but the only non-zero ones are

$$\Gamma^{1}_{22} = \lambda \sin\theta \cos\theta, \qquad \Gamma^{1}_{32} = \tfrac{1}{2}(\sin\theta)(1+\lambda),$$
$$\Gamma^{3}_{21} = -(1/2\sin\theta)(1-\lambda\cos^2\theta), \qquad \Gamma^{2}_{21} = \tfrac{1}{2}(\cot\theta)(1-\lambda),$$
$$\Gamma^{2}_{31} = (1/2\sin\theta)(1+\lambda), \qquad \Gamma^{3}_{31} = \tfrac{1}{2}(\cot\theta)(1+\lambda), \qquad \text{(A2.4)}$$

and those found by symmetry in the lower indices.

The Riemann tensor is given by

$$-R^{\mu}{}_{\gamma\alpha\beta} = \partial_{[\alpha}\Gamma^{\mu}{}_{\gamma\beta]} + \Gamma^{\mu}{}_{\sigma[\alpha}\Gamma^{\sigma}{}_{\gamma\beta]} \qquad \text{(A2.5)}$$

and we find

$$-R^{1}{}_{212} = \tfrac{1}{4}[1 - 3\lambda + \lambda(\cos^2\theta)(5+\lambda)],$$
$$-R^{3}{}_{232} = \tfrac{1}{4}[1 + \lambda + \lambda(1+\lambda)\cos^2\theta],$$
$$-R^{1}{}_{213} = \tfrac{1}{4}(1+\lambda)^2\cos\theta, \qquad -R^{2}{}_{223} = \tfrac{1}{4}(1+\lambda)^2\cos\theta,$$
$$-R^{1}{}_{313} = \tfrac{1}{4}(1+\lambda)^2, \qquad -R^{2}{}_{131} = \tfrac{1}{4}(1-3\lambda), \qquad -R^{3}{}_{131} = \tfrac{1}{4}(1+\lambda),$$
$$-R^{3}{}_{332} = \tfrac{1}{4}(1+\lambda)^2\cos\theta,$$
$$-R^{1}{}_{312} = \tfrac{1}{4}(1+\lambda)^2\cos\theta, \qquad \lambda = l_3^2/l_1^2 - 1. \qquad \text{(A2.6)}$$

The other components, except those that can be obtained from the above by symmetry, are zero.

The trace of the $R^{\mu}{}_{\gamma\alpha\beta}$ can be found. We obtain

$$R_1{}^1 = (1/2l_1^4)(2l_1^2 - l_3^2), \qquad R_2{}^2 = (1/2l_1^4)(2l_1^2 - l_3^2),$$
$$R_3{}^3 = l_3^2/2l_1^4, \qquad R_2{}^3 = (\lambda\cos\theta/l_1^2), \qquad R_3{}^2 = 0,$$
$$R = 2/l_1^2 - l_3^2/2l_1^4. \qquad \text{(A2.7)}$$

When computing the variation (31), we note that

$$\Box R_1{}^1 = -(1/4l_1^4)\lambda(1+\lambda), \qquad \nabla_\sigma\nabla^1 R_1{}^\sigma = (1/4l_1^4)\lambda(1+\lambda). \qquad \text{(A2.8)}$$

Naively, we might have expected that the derivatives of the Riemann tensor would be zero here, since from equation (A2.7) $R_1{}^1$ is a constant, but careful analysis refutes this expectation.

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